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Grassmann tensor network states and its renormalization for strongly correlated fermionic and bosonic states

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The projective construction (the slave-particle approach) has played an very important role in understanding strongly correlated systems, such as the emergence of fermions, anyons, and gauge theory in quantum spin liquids and quantum Hall states. Recently, fermionic Projected Entangled Pair States (fPEPS) have been introduced to efficiently represent many-body fermionic states. In this paper, we show that the strongly correlated bosonic/fermionic states obtained both from the projective construction and the fPEPS approach can be represented systematically as Grassmann tensor product states. This construction can also be applied to all other tensor network states approaches. The Grassmann tensor product states allow us to encode many-body bosonic/fermionic states efficiently with a polynomial number of parameters. We also generalize the tensor-entanglement renormalization group (TERG) method for complex tensor networks to Grassmann tensor networks. This allows us to approximate the norm and average local operators of Grassmann tensor product states in polynomial time, and hence leads to a variational approach for describing strongly correlated bosonic/fermionic systems in higher dimensions.

I. INTRODUCTION

Traditional condensed matter physics is based on two theories: symmetry breaking theory for phases and phase transitions, and Fermi liquid theory for metals. Within the Fermi liquid theory, one assumes that the ground state wave function for the electrons can be approximately described by a Slater determinant. In other words, one assumes that the many-electron ground state can be constructed by filling the single-particle energy levels. Such an energy-level filling picture becomes a foundation for traditional many-body physics. In this paper, we will call states obtained by filling single-particle energy levels as energy-level-filling (ELF) states. In addition to Slater determinant states, fermion paired states are also ELF states.

We may view the ELF construction of many-body states as an encoding method of a physically relevant subset of states. Although a random many-body state can only specified by an exponential amount of data, hence making it impossible to specify and calculate physical properties efficiently, physically relevant states seem to have a much simpler entanglement structure. A generic ELF state on a lattice can be written as

$$|\Psi_f\rangle = \exp\Big(\sum_{\langle ij\rangle} u_{ij} c_j^\dagger c_i^\dagger\Big)|0\rangle \tag{1}$$

where $\sum_{\langle ij\rangle}$ sums over the pairs of sites in the lattice. Here we consider a simple example of paired spinless fermions, and unpaired fermions can also be represented as limits of such states. Such a many-body fermionic state is encoded by polynomial amount information characterized by u_{ij}. A crucial property is that it is very easy to calculate the norm and the averages of any local operators for a ELF state; here "easy" means that the computational cost scales as a polynomial in the number of modes. This effective encoding and the ease of calculating physical quantities (such as energy) form the foundation of the standard mean-field theory for interacting electron systems. In such a ELF approach, we may view u_{ij} as variational parameters and minimize the average energy by varying u_{ij}. The $\bar{u}_{ij}$ that minimize the energy give us an approximated many-fermion ground state. From the form of $\bar{u}_{ij}$ we can determine which symmetry is broken and obtain the phase diagram of the system.

However, the ground states for some strongly correlated electron systems cannot be approximated by ELF states, i.e. they cannot be constructed by filling energy levels. One classic example is the filling fraction $\nu = 1/3$ Laughlin state[1]

$$\Psi_3 = \prod_{i<j}(z_i - z_j)^3 \mathrm{e}^{-\frac{1}{4}\sum_i |z_i|^2} \tag{2}$$

Although the $\nu = 1$ state $\Psi_1 = \prod_{i<j}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum_i |z_i|^2}$ is a ELF state, its cubic power, $\Psi_3 \sim (\Psi_1)^3$, is very different from any ELF states. In order to obtain the low energy effective theories for systems that cannot be described by ELF picture such as spin liquids and non-Fermi-liquid metallic states, a projective construction (also known as slave-particle approach or parton construction; see appendix B for a brief introduction) was developed[2–12]. Those states may appear in high T_c superconductors and other strongly interacting systems, and are by now widely used. Just like the ELF states, the projective states can also be characterized by a polynomial amount of information. So the projective approach can also be viewed as an efficient encoding of many-body states.

In the projective approach, we view the projective states as variational trial wave functions[9,11] for obtaining approximate ground state. What makes the projec-

tive approach so attractive is that from the form of the projective state, we can usually obtain the low energy effective theory that describes low energy excitations[8,11]. From the low energy effective theory, we have learned that projective states can capture many qualitatively new phenomena that ELF states fail to describe, such as fractional charge, fractional statistics and topological orders[7,8,12].

However, it is much harder to calculate the norm and local expectation values in the projective approach. Although expectation values can efficiently be calculated using variational Monte Carlo methods in the case that electron or spin operators can be expressed as the products of parton operators[13,14], Monte Carlo fails when those operators are expressed as sums of the products of parton operators[8,15,16]. As such there is no general efficient way to calculate the norm and the average of local operators.

In a different development, concepts of quantum information theory have allowed one to gain a better insight into the entanglement structure in ground states of generic Hamiltonians of strongly correlated quantum spins. It has been shown that ground states of local Hamiltonians obey so-called area laws, and that ground states can therefore be efficiently be represented by the class of so-called Matrix Product States in one dimension or Projected Entangled Pair States (PEPS) in higher dimensions[17,18]. This class of states share some resemblances with the states obtained in the projective approach, as they can be understood as projections of locally maximally entangled pairs. Importantly, techniques have been developed that make it possible to efficiently calculate expectation values of local operators for this class of states, as this can be done by contracting networks of tensor products. In this paper, we will show how to generalize such networks to include Grassmann tensors, and show how they can be contracted efficiently. This allows us to define a large and important subclass of the states obtained in the projective approach for which it is possible to calculate expectation values. It also allows us to generalize all tensor product state methods to the fermionic case; special subclasses include the recently introduced fermionic PEPS (fPEPS)[19] and the fermionic Multiscale Entanglement Renormalization Ansatz[20].

The Grassmann tensor product states allow us to express the norms of the projective wave functions in terms of Grassmann valued tensor network. Similarly, the average of any local operators for the projective wave functions can also be expressed in terms of Grassmann tensor networks with a few "impurity" tensors.

Throughout our study, we will also find that the Grassmann tensor product states can be more general than projective states. Using Grassman tensor networks, we can systematically construct more general strongly correlated states for both bosonic and fermionic systems. We only need polynomial amount of data to characterize the tensor network. Thus the Grassmann tensor network approach gives us an effective encoding for both fermionic and bosonic many-body states.

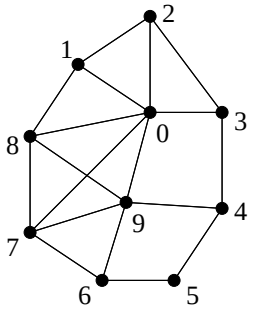

FIG. 1: A random lattice where the physical degrees of freedom are localized on the vertices.

Calculating the norm and the expectation value of local operators for a tensor product state can be exponentially hard in general[21]. However, many possible polynomial approximation schemes, including the tensor-entanglement renormalization group (TERG) method, have been proposed in recent years[22–27]. In this paper, we will show how the TERG method[24] can also be applied to Grassmann tensor network. In particular, this implies that the average energy and other physical quantities of a Grassmann tensor network state in two dimensions can be calculated efficiently by using tensor network renormalization approach. This Grassmann tensor network approach is the natural approach for expressing fermions in tensor network methods and hence provides a new starting point for studying strongly correlated bosonic/fermionic systems.

II. TENSOR-NETWORK REPRESENTATION OF ELF STATES

Let us first consider a simple example of using tensor networks to represent a ELF state on an arbitrary graph, where the fermions live on the vertices of the graph (see Fig. 1):

$$|\Psi_f\rangle = \exp\left[\sum_{ij} u_{ij} c_j^\dagger c_i^\dagger\right]|0\rangle = \prod_{ij}\left(1 + u_{ij} c_j^\dagger c_i^\dagger\right)|0\rangle \quad (3)$$

As $u_{ij}c_j^\dagger c_i^\dagger = -u_{ij}c_i^\dagger c_j^\dagger$, w.l.o.g. $u_{ij} = -u_{ji}$. The many-body wave function $\Psi_f(\{m_i\})$ is given by

$$\Psi_f(\{m_i\}) = \langle 0|\prod_i (c_i)^{m_i}\prod_{ij}\left(1 + u_{ij} c_j^\dagger c_i^\dagger\right)|0\rangle \quad (4)$$

where $m_i = 0, 1$ indicating if the site-i is empty or occupied. Note that the fermion operators c_i in the product $\prod_i (c_i)^{m_i}$ is ordered in the following way

$$\prod_i (c_i)^{m_i} \equiv (c_1)^{m_1}(c_2)^{m_2}(c_3)^{m_3}... \quad (5)$$

Motivated by fermionic path integral, we may represent the above wave function in terms of Grassmann numbers θ_i and their derivatives $\mathrm{d}\theta_i$, which satisfy

$$\begin{aligned} \theta_i\theta_j &= -\theta_j\theta_i, & \mathrm{d}\theta_i\,\mathrm{d}\theta_j &= -\mathrm{d}\theta_j\,\mathrm{d}\theta_i, \\ \int \mathrm{d}\theta_i\theta_j &= \delta_{ij} & \int \mathrm{d}\theta_i 1 &= 0. \end{aligned} \tag{6}$$

We find that the wave function can be rewritten as

$$\begin{aligned} \Psi_f(\{m_i\}) &= \int \prod_i T_i^{m_i} \prod_{ij} G_{ij}, \\ T_i^1 &= \mathrm{d}\theta_i, \quad T_i^0 = 1, \quad G_{ij} = 1 + u_{ij}\theta_j\theta_i, \end{aligned} \tag{7}$$

where $\int$ "integrates out" all Grassmann numbers.

Similarly, $\Psi_f^*(\{m_i\})$ can be expressed as

$$\begin{aligned} \Psi_f^*(\{m_i\}) &= \int \prod_i \bar{T}_i^{m_i} \prod_{ij} (1 + u_{ij}^*\bar{\theta}_j\bar{\theta}_i), \\ &= \int \widetilde{\prod_i} \bar{T}_i^{m_i} \prod_{ij} \bar{G}_{ij}, \\ \bar{T}_i^1 &= \mathrm{d}\bar{\theta}_i, \quad \bar{T}_i^0 = 1, \\ \bar{G}_{ij} &= 1 - u_{ij}^*\bar{\theta}_j\bar{\theta}_i = 1 + u_{ij}^*\bar{\theta}_i\bar{\theta}_j. \end{aligned} \tag{8}$$

Note that $\prod_i$ and $\widetilde{\prod_i}$ have different orders:

$$\prod_i T_i^{m_i} \equiv T_1^{m_1} T_2^{m_2} T_3^{m_3} ... \qquad \widetilde{\prod_i} T_i^{m_i} \equiv ... T_3^{m_3} T_2^{m_2} T_1^{m_1}. \tag{9}$$

Here we have used the following identity

$$\int [\prod_i (\mathrm{d}\theta_i)^{m_i}][\theta_{i_1}\theta_{i_2}...] = \int [\widetilde{\prod_i} (\mathrm{d}\theta_i)^{m_i}][...\theta_{i_2}\theta_{i_1}] \tag{10}$$

Thus the norm of the wave function is given by

$$\begin{aligned} \langle\Psi_f|\Psi_f\rangle &= \sum_{\{m_i\}} \int (\widetilde{\prod_i} \bar{T}_i^{m_i} \prod_{ij} \bar{G}_{ij})(\prod_i T_i^{m_i} \prod_{ij} G_{ij}) \\ &= \sum_{\{m_i\}} \int \prod_i \bar{T}_i^{m_i} T_i^{m_i} \prod_{ij} \bar{G}_{ij} G_{ij} \\ &= \int \prod_i \boldsymbol{T}_i \prod_{ij} \boldsymbol{G}_{ij} \end{aligned} \tag{11}$$

where

$$\boldsymbol{T}_i = 1 + \mathrm{d}\bar{\theta}_i\mathrm{d}\theta_i,\ \boldsymbol{G}_{ij} = (1 + u_{ij}^*\bar{\theta}_i\bar{\theta}_j)(1 + u_{ij}\theta_j\theta_i). \tag{12}$$

We may view $\boldsymbol{T}_i$ as a dimension-1 tensor on the vertex i, and $\boldsymbol{G}_{ij}$ as a dimension-1 rank-2 tensor on the link ij. Then $\prod_i \boldsymbol{T}_i^{m_i} \prod_{ij} \boldsymbol{G}_{ij}$ can be viewed as the tensor trace on such a tensor network. Note that the tensors contain Grassmann numbers and $\boldsymbol{T}_i$'s always appear in front of $\boldsymbol{G}_{ij}$'s. We see that the norm of a fermion wave function can be expressed as the tensor trace of a Grassmann tensor network.

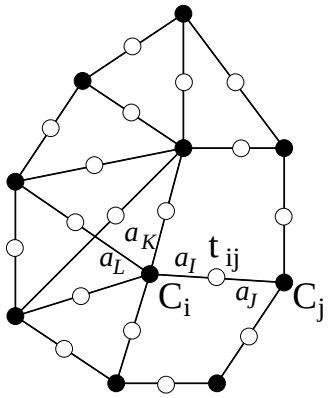

FIG. 2: A strongly correlated system on a graph. The physical degrees of freedom live on the vertices. The vertices (the filled dots) are labeled by i, j, *etc* . The open dots are labeled by ij, *etc* . The links connecting open and filled dots are labeled by I, J, K, *etc* .

III. GRASSMANN TENSOR-NETWORK REPRESENTATION OF GENERIC STRONGLY CORRELATED FERMIONIC AND BOSONIC STATES

The above result can be generalized to strongly correlated fermionic states as well as strongly correlated bosonic states. Let us assume the physical degrees of freedom are localized on the vertices of a graph (see Fig. 1). The vertices are labeled by i, j, *etc* . The states on vertex i are label by $m_i = 1, 2, ...$ Let a sign function $s(m_i)$ to have the following property. $s(m_i) = 1$ if the state m_i is a bosonic state and $s(m_i) = -1$ if the state m_i is a fermionic state. To construct a many-body wave function $\Psi(\{m_i\})$, we introduce some fermion operators ψ_i^α on each vertex i with α denotes the fermion species(such as spin). We also use ij *etc* to label the open dots and I, J, *etc* to label the links between the open and filled dots in Fig. 2. Then we can construct $\Psi(\{m_i\})$ as

$$\Psi(\{m_i\}) = \sum_{\{a_I\}} \langle 0| \prod_i C_{m_i;a_K a_L ...} \prod_{\langle ij\rangle} t_{ij;a_I a_J} |0\rangle\,, \tag{13}$$

where K, L, *etc* in $C_{m_i;a_K a_L ...}$ label the links that connect to the vertex i, and I, J in $t_{ij;a_I a_J}$ label the links that connect the vertex i, j with the open dot ij. All the link indices $a_K, a_L, a_I, a_J ... = 1, ..., D$ are the inner indices defined in the standard (bosonic) tesor product states(TPS). Here $t_{ij;a_I b_J}$ only contains $\psi_i^{\alpha\dagger}$ and $\psi_j^{\alpha\dagger}$ operators and $C_{m_i;a_K a_L ...}$ only contains ψ_i^α operators. $t_{ij;a_I a_J}$ always contains even numbers of fermion operators. $C_{m_i;a_K a_L ...}$ contains an even numbers of fermion operators if $s(m_i) = 1$ and contains odd numbers of fermion operators if $s(m_i) = -1$.

Introducing the Grassmann numbers θ_i^α for each ver-

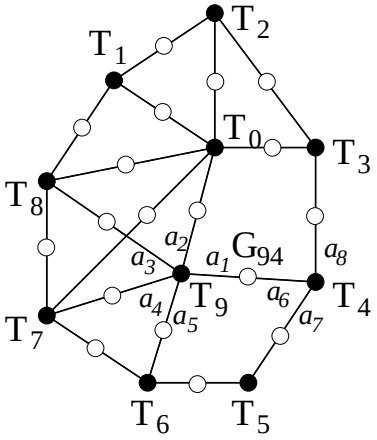

FIG. 3: A tensor network formed by two kinds of Grassmann tensors $T^{m_i}_{i;a_K a_{L\cdots}}$ and $G_{ij;a_I a_J}$ *etc* . The filled dots (the vertices) are labeled by i which represent $T^{m_i}_{i;a_K a_{L\cdots}}$, and the open dots are labeled by ij which represent $G_{ij;a_I a_J}$. The lines (labeled by I, J, *etc*) connecting the dots represent the indices a_I of the tensors. For two tenors connected by a line, the values of the associated indices in the two tensor are set to be equal, and those values are summed over in the tensor trace. Such a summation is represented by $\sum_{\{a_I\}}$ in (13).

tex, we can express the many-body wave function as

$$\Psi(\{m_i\}) = \sum_{\{a_I\}} \int \prod_i T^{m_i}_{i;a_K a_{L\cdots}} \prod_{ij} G_{ij;a_I a_J}, \qquad (14)$$

where $T^{m_i}_{i;a_K a_{L\cdots}}$ is obtained from $C_{m_i;a_K a_{L\cdots}}$ by replacing ψ^α_i by $\mathrm{d}\theta^\alpha_i$ and $G_{ij;a_I a_J}$ is obtained from $t_{ij;a_I a_J}$ by replacing $\psi^{\alpha\dagger}_i$ by θ^α_i. So, $\Psi(\{m_i\})$ can be expressed as a tensor trace over a Grassmann tensor network (see Fig. 3).

Similarly $\Psi^*(\{m_i\})$ can be expressed as

$$\Psi^*(\{m_i\}) = \sum_{\{a_I\}} \int \widetilde{\prod_i} \bar{T}^{m_i}_{i;a_K a_{L\cdots}} \prod_{ij} \bar{G}_{ij;a_I a_J}, \qquad (15)$$

where $\bar{T}^{m_i}_{i;a_K a_{L\cdots}}$ is obtained from $C^\dagger_{m_i;a_K a_{L\cdots}}$ by replacing $\psi^{\alpha\dagger}_i$ by $\mathrm{d}\bar{\theta}^\alpha_i$ and $\bar{G}_{ij;a_I a_J}$ is obtained from $t^\dagger_{ij;a_I a_J}$ by replacing ψ^α_i by $\bar{\theta}^\alpha_i$. Essentially, (14) is a fermionic generalization of the standard (bosonic) tensor product states(TPS). If there is no Grassmann number in T, G, it becomes the standard TPS (one can further put $G_{ij;a_I a_J} = \delta_{a_I,a_J}$ if the tensor contraction of inner indices are made over a trivial metric).

The norm of the wave function can be calculated in

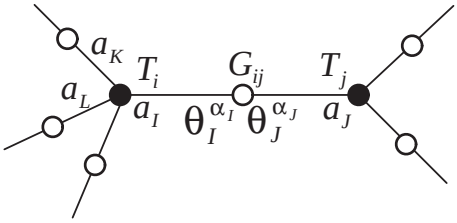

FIG. 4: the Grassmann number θ^α_I is associated with the link I that connects to vertex i. The tensor G_{ij} contains only Grassmann numbers θ^α_I and θ^α_J.

the same way:

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \sum_{\{m_i\},\{a_I\bar{a}_I\}} \int (\widetilde{\prod_i} \bar{T}^{m_i}_{i;\bar{a}_K \bar{a}_{L\cdots}} \prod_{ij} \bar{G}_{ij;\bar{a}_I \bar{a}_J}) \times \\ &\qquad (\prod_i T^{m_i}_{i;a_K a_{L\cdots}} \prod_{ij} G_{ij;a_I a_J}) \\ &= \sum_{\{m_i\},\{a_I\bar{a}_I\}} \int \prod_i \bar{T}^{m_i}_{i;\bar{a}_K \bar{a}_{L\cdots}} T^{m_i}_{i;a_K a_{L\cdots}} \prod_{ij} \bar{G}_{ij;\bar{a}_I \bar{a}_J} G_{ij;a_I a_J} \\ &= \sum_{\{a_I\bar{a}_I\}} \int \prod_i \boldsymbol{T}_{i;a_K \bar{a}_K, a_L \bar{a}_{L\cdots}} \prod_{ij} \boldsymbol{G}_{ij;a_I \bar{a}_I, a_J \bar{a}_J} \qquad (16)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{T}_{i;a_K \bar{a}_K, a_L \bar{a}_{L\cdots}} &= \sum_m \bar{T}^m_{i;\bar{a}_K \bar{a}_{L\cdots}} T^m_{i;a_K a_{L\cdots}} \\ \boldsymbol{G}_{ij;a_I \bar{a}_I, a_J \bar{a}_J} &= \bar{G}_{ij;\bar{a}_I \bar{a}_J} G_{ij;a_I a_J}. \qquad (17)\end{aligned}$$

Again, the norm is a tensor trace of a Grassmann tensor network (see Fig. 3 where T, G are replaced by $\boldsymbol{T}$, $\boldsymbol{G}$ and each link is indexed by a pair $a_I \bar{a}_I$). We may combine the pair of indices $(a_I \bar{a}_I)$ into one p_I, and rewrite the above as:

$$\langle\Psi|\Psi\rangle = \sum_{\{p_I\}} \int \prod_i \boldsymbol{T}_{i;p_K p_{L\cdots}} \prod_{ij} \boldsymbol{G}_{ij;p_I p_J} \qquad (18)$$

IV. GRASSMAN TENSOR PRODUCT STATES – A "BOND" FORM

Although the Grassmann tensor network representations for the norm of fermion wave function, (11) and (18), are simple and compact, however, it is not easy to implement the renormalization calculation for such a generic Grassmann tensor network. In this section, we will consider Grassmann tensor networks that have a special form which makes the renormalization calculation easier. Although the Grassmann tensor networks have a special form, they can still represent generic strongly correlated fermionic states described by the more generic form (18).

In the previous Grassmann tensor network, the Grassmann numbers on vertex i, $\theta^{\alpha_i}_i$, are labeled by $\alpha_i =$

$1, 2, ...$ To construct the more special form of the Grassmann tensor network, we separate the Grassmann numbers on vertex i into several groups, one group for each link that connects the vertex i. So each group of Grassmann numbers is actually associated with a link labeled by I. Thus it is more convenient to relabel the Grassmann numbers on vertex i as $\theta_I^{\alpha_I}$, which correspond to the Grassmann numbers on vertex i and associated with the link I that connects to the vertex (see Fig. 4).

With the new labeling scheme, we can specify the special form of the Grassmann tensor network by requiring $G_{ij;a_Ia_J}$ to only contain the Grassmann numbers θ_I^α and θ_J^α (see Fig. 4). (Note that $G_{ij;a_Ia_J}$ still contain even numbers of θ_I^α and θ_J^α.) Under such scheme, the wavefunction (14) can be represented as:

$$\Psi(\{m_i\}) = \sum_{\{a_I\}} \int \prod_i T^{m_i}_{i;a_Ka_L...} \prod_{ij} G_{ij;a_Ia_J}, \tag{19}$$

where

$$\begin{aligned} T^{m_i}_{i;a_Ka_L...} &= \sum_{\{l_K^{\alpha_K}\}\{l_L^{\alpha_L}\}..} \mathrm{T}^{m_i;\{l_K^{\alpha_K}\}\{l_L^{\alpha_L}\}..}_{i;a_Ka_L..} \prod_{I\in i}\widetilde{\prod_{\alpha_I}}(\mathrm{d}\theta_I^{\alpha_I})^{l_I^{\alpha_I}} \\ G_{ij;a_Ia_J} &= \sum_{\{l_I^{\alpha_I}\}\{l_J^{\alpha_J}\}} \mathrm{G}^{\{l_I^{\alpha_I}\}\{l_J^{\alpha_J}\}}_{ij;a_Ia_J} \prod_{\alpha_J}(\theta_J^{\alpha_J})^{l_J^{\alpha_J}} \prod_{\alpha_I}(\theta_I^{\alpha_I})^{l_I^{\alpha_I}} \end{aligned} \tag{20}$$

Here $I \in i$ means the links that connect to vertex i. Note that, in the expression of G_{ij}, we have assumed that the link I connects vertex i with the open dot ij, and the link J connects vertex j with the open dot ij (see Fig. 4). We also note that $l_K^{\alpha_K} = 0$ or 1 indicates the presence or the absence of the Grassmann number $\mathrm{d}\theta_K^{\alpha_K}$. So we may interpret $l_K^{\alpha_K}$ as an "occupation number of fermions".Here $\sum_{\{l_K^{\alpha_K}\}\{l_L^{\alpha_L}\}...}$ sums over all the possible "occupation" distributions $\{l_K^{\alpha_K} = 0, 1|\alpha_K = 1, 2, ...\}$, $\{l_L^{\alpha_L} = 0, 1|\alpha_L = 1, 2, ...\}$,... Similarly as in (14), $\sum_{I\in i}\sum_{\alpha_I} l_I^{\alpha_I} =$ odd represents a fermionic state and $\sum_{I\in i}\sum_{\alpha_I} l_I^{\alpha_I} =$ even represents a bosonic state. However, $\sum_{\alpha_I} l_I^{\alpha_I} + \sum_{\alpha_J} l_J^{\alpha_J}$ should always to be even. Such type of representations for fermionic wave functions was first introduced in Ref. 19.

In the new form, the norm of wave function can be expressed in the same way as in (18) with:

$$\begin{aligned} \boldsymbol{T}_{i;a_K\bar{a}_K,a_L\bar{a}_L...} &= \sum_m \bar{T}^m_{i;\bar{a}_K\bar{a}_L...} T^m_{i;a_Ka_L...} \\ \boldsymbol{G}_{ij;a_I\bar{a}_I,a_J\bar{a}_J} &= \bar{G}_{ij;\bar{a}_I\bar{a}_J} G_{ij;a_Ia_J}. \end{aligned} \tag{21}$$

Here $\bar{T}^m_{i;\bar{a}_K\bar{a}_L...}$ and $\bar{G}_{ij;\bar{a}_I\bar{a}_J}$ are defined as:

$$\begin{aligned} \bar{T}^{m_i}_{i;\bar{a}_K\bar{a}_L...} &= \sum_{\{\bar{l}_K^{\alpha_K}\}\{\bar{l}_L^{\alpha_L}\}..} \left[\mathrm{T}^{m_i;\{\bar{l}_K^{\alpha_K}\}\{\bar{l}_L^{\alpha_L}\}..}_{i;\bar{a}_K\bar{a}_L..}\right]^* \prod_{I\in i}\prod_{\alpha_I}(\mathrm{d}\bar{\theta}_I^{\alpha_I})^{\bar{l}_I^{\alpha_I}} \\ \bar{G}_{ij;\bar{a}_I\bar{a}_J} &= \sum_{\{\bar{l}_I^{\alpha_I}\}\{\bar{l}_J^{\alpha_J}\}} \left[\mathrm{G}^{\{\bar{l}_I^{\alpha_I}\}\{\bar{l}_J^{\alpha_J}\}}_{ij;\bar{a}_I\bar{a}_J}\right]^* \widetilde{\prod_{\alpha_I}}(\bar{\theta}_I^{\alpha_I})^{\bar{l}_I^{\alpha_I}} \widetilde{\prod_{\alpha_J}}(\bar{\theta}_J^{\alpha_J})^{\bar{l}_J^{\alpha_J}} \end{aligned} \tag{22}$$

By combining the pair of indices $(a_I\bar{a}_I)$ into one p_I and reordering those $d\theta_I^{\alpha_I}, d\bar{\theta}_I^{\alpha_I}(\theta_I^{\alpha_I}, \bar{\theta}_I^{\alpha_I})$, the tensors $\boldsymbol{T}$ and $\boldsymbol{G}$ in (18) can be further expanded as

$$\begin{aligned} \boldsymbol{T}_{i;p_Kp_L...} &= \sum_{\{n_K^{\alpha_K}\}\{n_L^{\alpha_L}\}..} \mathcal{T}^{\{n_K^{\alpha_K}\}\{n_L^{\alpha_L}\}..}_{i;p_Kp_L..} \prod_{I\in i}\widetilde{\prod_{\alpha_I}}(\mathrm{d}\eta_I^{\alpha_I})^{n_I^{\alpha_I}} \\ \boldsymbol{G}_{ij;p_Ip_J} &= \sum_{\{n_I^{\alpha_I}\}\{n_J^{\alpha_J}\}} \mathcal{G}^{\{n_I^{\alpha_I}\}\{n_J^{\alpha_J}\}}_{ij;p_Ip_J} \prod_{\alpha_J}(\eta_J^{\alpha_J})^{n_J^{\alpha_J}} \prod_{\alpha_I}(\eta_I^{\alpha_I})^{n_I^{\alpha_I}} \end{aligned} \tag{23}$$

where the group of Grassmann numbers $\{\eta_I^{\alpha_I}|\alpha_I = 1, 2, ...\}$ is the combination of $\{\theta_I^{\beta_I}|\beta_I = 1, 2, ...\}$ and $\{\bar{\theta}_I^{\beta_I}|\beta_I = 1, 2, ...\}$. Notice $\mathcal{T}^{\{n_K^{\alpha_K}\}\{n_L^{\alpha_L}\}...}_{i;p_Kp_L...}$ vanishes if the total numbers of Grassmann numbers $\mathrm{d}\eta_I^\alpha$ is odd and $\mathcal{G}^{\{n_I^{\alpha_I}\}\{n_J^{\alpha_J}\}}_{ij;p_Ip_J}$ vanishes if the total numbers of Grassmann numbers $\eta_I^\alpha, \eta_J^\alpha$ is odd:

$$\begin{aligned} \mathcal{T}^{\{n_K^{\alpha_K}\}\{n_L^{\alpha_L}\}...}_{i;p_Kp_L...} = 0, \ \text{ if } \sum_{I\in i}\sum_{\alpha_I} n_I^{\alpha_I} = \text{ odd}, \\ \mathcal{G}^{\{n_I^{\alpha_I}\}\{n_J^{\alpha_J}\}}_{ij;p_Ip_J} = 0, \ \text{ if } \sum_{\alpha_I} n_I^{\alpha_I} + \sum_{\alpha_J} n_J^{\alpha_J} = \text{ odd}. \end{aligned} \tag{24}$$

To gain some intuitive understanding of the new form of the Grassmann tensor network, let us represent the free fermion wave function Ψ_f (4) using the new form of Grassmann tensor network:

$$\begin{aligned} \Psi_f(\{m_i\}) &= \int \prod_i T_i^{m_i} \prod_{ij} G_{ij}, \\ T_i^1 = \sum_{I\in i} \mathrm{d}\theta_I, \quad T_i^0 = 1, \quad G_{ij} &= 1 + u_{ij}\theta_J\theta_I, \end{aligned} \tag{25}$$

where $\sum_{I\in i}$ sums over all the links that connect to vertex i. The norm of such a free fermion state is given by

$$\begin{aligned} \langle\Psi_f|\Psi_f\rangle &= \int \prod_i \boldsymbol{T}_i \prod_{ij} \boldsymbol{G}_{ij} \\ \boldsymbol{T}_i &= 1 + [\sum_{K\in i} \mathrm{d}\bar{\theta}_K][\sum_{L\in i} \mathrm{d}\theta_L], \\ \boldsymbol{G}_{ij} &= (1 + u^*_{ij}\bar{\theta}_I\bar{\theta}_J)(1 + u_{ij}\theta_J\theta_I). \end{aligned} \tag{26}$$

Again, in the expression of $\boldsymbol{G}_{ij}$, we have assumed that the link I connects the vertex i with the open dot ij, and the link J connects the vertex j with the open dot ij.

We would like to stress that the above formulism is very general. It can be used to express free fermion

states, as well as strongly correlated fermionic/bosonic states obtained from the projective construction. It can even express strongly correlated fermionic/bosonic states beyond the projective construction. We also would like to stress that the graphs discussed in this paper do not have to be a real space lattice.

V. STRONGLY CORRELATED STATES FROM THE PROJECTIVE CONSTRUCTION

As we have mentioned, the Grassmann tensor network can represent very general strongly correlated fermionic/bosonic states. In this section, we will concentrate on strongly correlated fermionic/bosonic states obtained from a projective construction and show that all of these states can be expressed as Grassmann tensor product states. The Grassmann tensor network for such projective states takes a particularly simple form.

To construct the Grassmann tensor network for the projective states, let us first construct the Grassmann tensor network for a ELF state:

$$|\Psi_f\rangle = \exp\left[\sum_{\langle ij\rangle}(u_{ij})_{\alpha\beta}\psi^\dagger_{j,\beta}\psi^\dagger_{i,\alpha}\right]|0\rangle \tag{27}$$

where there can be several kinds of fermions labeled by α,β on each vertex. As discussed in section IV, the above ELF state can be expressed in terms of Grassmann tensor product state:

$$\begin{aligned}\Psi_f(\{m_{i,\alpha}\}) &= \langle 0|\prod_i\prod_\alpha(\psi_{i,\alpha})^{m_{i,\alpha}}|\Psi_f\rangle\\ &= \int\prod_i T_i^{\{m_{i,\alpha}\}}\prod_{ij}G_{ij},\\ T_i^{\{m_{i,\alpha}\}} &= \prod_\alpha(\sum_{I\in i}\mathrm{d}\theta_I^\alpha)^{m_{i,\alpha}}\\ G_{ij} &= \exp\left[\sum_{\langle ij\rangle}(u_{ij})_{\alpha\beta}\theta_J^\beta\theta_I^\alpha\right]\end{aligned} \tag{28}$$

where $m_{i,\alpha}=0,1$ describes the occupation of the α^{th} fermion on vertex i. $T_i^{\{m_{i,\alpha}\}}$ is obtained from $\prod_\alpha(\psi_{i,\alpha})^{m_{i,\alpha}}$ by replacing $\psi_{i,\alpha}$ by $\sum_{I\in i}\mathrm{d}\theta_I^\alpha$ where $\sum_{I\in i}$ sums over all the links that connects to the vertex i. G_{ij} is obtained from $\exp\left[\sum_{\langle ij\rangle}(u_{ij})_{\alpha\beta}\psi^\dagger_{j,\beta}\psi^\dagger_{i,\alpha}\right]$ by replacing $\psi^\dagger_{j,\beta}\psi^\dagger_{i,\alpha}$ by $\theta_J^\beta\theta_I^\alpha$ where I and J label the links that connects to the open dot ij between the two vertices i and j. Note that (28) is a generalization of (25).

To obtain a strongly correlated fermionic state from the above ELF state, for example, we may assume $\alpha=1,2,3$ and consider the following many-body wave function:

$$\Psi_{\text{corr}}(\{m_i\}) = \langle 0|\prod_i(c_i)^{m_i}|\Psi_f\rangle,\quad c_i=\prod_{\alpha=1}^3\psi_{i,\alpha}, \tag{29}$$

where $m_i=0,1$. We note that $\Psi_{\text{corr}}(\{m_i\})$ is a many-body wave function for a spinless fermion system. It is a strongly correlated projective state which is very different from any ELF state. Such a state can be expressed in terms of Grassmann tensor network

$$\Psi_{\text{corr}}(\{m_i\}) = \int\prod_i T_i^{m_i}\prod_{ij}G_{ij}, \tag{30}$$

$$T_i^{m_i} = \Big(\prod_{\alpha=1}^3\sum_{I\in i}\mathrm{d}\theta_I^\alpha\Big)^{m_i},\quad G_{ij}=\exp\left[\sum_{\langle ij\rangle}(u_{ij})_{\alpha\beta}\theta_J^\beta\theta_I^\alpha\right]$$

Similarly, to obtain a strongly correlated hardcore bosonic state (such as spin liquid state) using projective construction, for example, we may assume $\alpha=1,2$ and consider the following many-body wave function:

$$\Psi_{\text{spin}}(\{m_i\}) = \langle 0|\prod_i(b_i)^{m_i}|\Psi_f\rangle,\quad b_i=\psi_{i,1}\psi_{i,2}, \tag{31}$$

where $m_i=0,1$. $\Psi_{\text{spin}}(\{m_i\})$ is a many-body wave function for a hardcore boson system. It can also be viewed as a wave function for a spin-1/2 system. Such a state can be expressed in terms of Grassmann tensor network

$$\Psi_{\text{spin}}(\{m_i\}) = \int\prod_i T_i^{m_i}\prod_{ij}G_{ij}, \tag{32}$$

$$T_i^{m_i} = \Big(\prod_{\alpha=1}^2\sum_{I\in i}\mathrm{d}\theta_I^\alpha\Big)^{m_i},\quad G_{ij}=\exp\left[\sum_{\langle ij\rangle}(u_{ij})_{\alpha\beta}\theta_J^\beta\theta_I^\alpha\right].$$

Both strongly correlated states $\Psi_{\text{corr}}(\{m_i\})$ and $\Psi_{\text{spin}}(\{m_i\})$ are parameterized by $(u_{ij})_{\alpha\beta}$. We may view $(u_{ij})_{\alpha\beta}$ as variational parameters. After finding $(\bar{u}_{ij})_{\alpha\beta}$ that minimize the average energy, we obtain the approximated ground state. We can also obtain the low energy effective theory from the form of the ansatz $(\bar{u}_{ij})_{\alpha\beta}$.

Finally we would like to point out that the ways to construct ELF states and projective states discussed in sections II and V are simple but not efficient. The Grassmann tensor product states derived in that way will usually contain long range connections, which is not necessary in special cases[19].

VI. FERMION COHERENT STATE REPRESENTATION

In above sections, we have represented the Grassmann tensor network wavefunctions under the Fock basis. Although the Fock basis representation is simple and straightforward to derive, however, because of the anticommutating relations for different Grassmann numbers, the wavefunctions depend on the ordering the local Grassmann tensors $T^{m_i}_{i;a_Ka_L\cdots}$ and are inconvenient for simulating physical quantities for fermion systems.

In this section, we would like to introduce the fermion coherent state representation for Grassmann tensor product states. We show the fermion wavefunctions in this

basis are independent of the ordering of local Grassmann tensors. To see this explicitly, let us consider a simple spinless fermion tensor product state:

$$|\Psi\rangle = \sum_{\{m_i\}} \sum_{\{a_I\}} \int \prod_i [c_i^\dagger]^{m_i} T^{m_i}_{i;a_K a_L \ldots} \prod_{ij} G_{ij;a_I a_J} |0\rangle, \tag{33}$$

where $m_i = 0, 1$ represents the fermion occupation numbers. It is easy to check that we can derive the Grassmann tensor product wavefunction (19) under the following Fock basis:

$$\prod_i (c_i)^{m_i} |0\rangle \equiv (c_1^\dagger)^{m_1} (c_2^\dagger)^{m_2} (c_3^\dagger)^{m_3} \ldots |0\rangle \tag{34}$$

The over complete fermion coherent state basis is defined as:

$$|\eta\rangle \equiv \prod_i (1 - \eta_i c_i^\dagger)|0\rangle, \tag{35}$$

with closure relation:

$$\int \prod_i \mathrm{d}\eta_i^* \, \mathrm{d}\eta_i (1 - \eta_i^* \eta_i) |\eta\rangle\langle\eta| = 1. \tag{36}$$

It is easy to derive the wavefunction for Grassmann tensor product state (33) under such a basis:

$$\langle\eta|\Psi\rangle = \sum_{\{m_i\}} \sum_{\{a_I\}} \int \prod_i \eta_i^{*m_i} T^{m_i}_{i;a_K a_L \ldots} \prod_{ij} G_{ij;a_I a_J}, \tag{37}$$

If we redefine the Grassmann tensor $T^{m_i}_{i;a_K a_L \ldots}$ as:

$$T^{m_i}_{i;a_K a_L \ldots} = \sum_{\{l_K^{\alpha_K}\}\{l_L^{\alpha_L}\}_{..}} \mathrm{T}^{m_i;\{l_K^{\alpha_K}\}\{l_L^{\alpha_L}\}_{..}}_{i;a_K a_L ..} \eta_i^{*m_i} \prod_{I \in i} \widetilde{\prod_{\alpha_I}} (\mathrm{d}\theta_I^{\alpha_I})^{l_I^{\alpha_I}} \tag{38}$$

then the wavefunction can be represented as:

$$\Psi_{\mathrm{coh}}(\eta^*) = \sum_{\{m_i\},\{a_I\}} \int \prod_i T^{m_i}_{i;a_K a_L \ldots} \prod_{ij} G_{ij;a_I a_J}, \tag{39}$$

Notice under the new definition, $T^{m_i}_{i;a_K a_L \ldots}$ always contain *even* number of Grassmann numbers hence the definition of the wavefunction (39) is independent of how we order those local Grassmann tensors $T^{m_i}_{i;a_K a_L \ldots}$.

It is very convenient to use the above wavefunction to calculate local physical quantities. The only thing we need to take care is the over complete nature of the basis, hence a proper measure is needed when we calculate the inner product for a wavefunction. For example, the norm of the wavefunction (39) can be calculated as:

$$\begin{aligned} &\int \prod_i \mathrm{d}\eta_i^* \, \mathrm{d}\eta_i (1 - \eta_i^* \eta_i) \Psi^*_{\mathrm{coh}}(\eta) \Psi_{\mathrm{coh}}(\eta^*) \\ &= \sum_{\{p_I\}} \int \prod_i \boldsymbol{T}_{i;p_K p_L \ldots} \prod_{ij} \boldsymbol{G}_{ij;p_I p_J}, \end{aligned} \tag{40}$$

which is exactly the same result as we derived before(tensors $\boldsymbol{T}$ and $\boldsymbol{G}$ are defined in Eq. (21)).

Other physical quantities like the average energy can also be easily calculated in a similar way. For any local operators containing $c^\dagger$ and c, we only need to replace $c^\dagger$ with η^* and c with η, then integrate out the Grassmann number respect to a proper measure as we do for calculating the norm. For example, the nearest neighbor paring term $c_i^\dagger c_j^\dagger$ can be expressed as:

$$\begin{aligned} \langle\Psi|c_i^\dagger c_j^\dagger|\Psi\rangle &= \int \mathrm{d}\eta_i^* \, \mathrm{d}\eta_i (1 - \eta_i^* \eta_i) \, \mathrm{d}\eta_j^* \, \mathrm{d}\eta_j (1 - \eta_j^* \eta_j) \prod_{i'} \mathrm{d}\eta_{i'}^* \, \mathrm{d}\eta_{i'} (1 - \eta_{i'}^* \eta_{i'}) \eta_i^* \eta_j^* \Psi^*_{\mathrm{coh}}(\eta) \Psi_{\mathrm{coh}}(\eta^*) \\ &= - \sum_{a_M \bar{a}_M .. a_K \bar{a}_K .. \{m_{i'}\},\{a_{I'} \bar{a}_{I'}\}} \int \bar{T}^1_{j;\bar{a}_M \bar{a}_N ..} \bar{T}^1_{i;\bar{a}_K \bar{a}_L ..} \bar{G}_{ij;\bar{a}_I \bar{a}_J} T^0_{i;a_K a_L ..} T^0_{j;a_M a_N ..} G_{ij;a_I a_J} \\ &\times \widetilde{\prod_{i'}} \bar{T}^{m_{i'}}_{i';\bar{a}_{K'} \bar{a}_{L'} ..} \prod_{i'j'} \bar{G}_{i'j';\bar{a}_{I'} \bar{a}_{J'}} \prod_{i'} T^{m_{i'}}_{i';a_{K'} a_{L'} ..} \prod_{i'j'} G_{i'j';a_{I'} a_{J'}} \\ &= \sum_{\{a_{I'} \bar{a}_{I'}\}} \int \boldsymbol{T}'_{i;a_K \bar{a}_K, a_L \bar{a}_L \ldots} \boldsymbol{T}''_{j;a_M \bar{a}_M, a_N \bar{a}_N \ldots} \boldsymbol{G}_{ij;a_I \bar{a}_I, a_J \bar{a}_J} \prod_{i'} \boldsymbol{T}_{i';a_{K'} \bar{a}_{K'}, a_{L'} \bar{a}_{L'} \ldots} \prod_{i'j'} \boldsymbol{G}_{i'j';a_{I'} \bar{a}_{I'}, a_{J'} \bar{a}_{J'}}, \end{aligned} \tag{41}$$

where the impurity tensors $\boldsymbol{T}'$ and $\boldsymbol{T}''$ are defined as:

$$\begin{aligned} \boldsymbol{T}'_{i;a_K \bar{a}_K, a_L \bar{a}_L \ldots} &= \bar{T}^1_{i;\bar{a}_K \bar{a}_L \ldots} T^0_{i;a_K a_L \ldots} \\ \boldsymbol{T}''_{j;a_M \bar{a}_M, a_N \bar{a}_N \ldots} &= \bar{T}^1_{j;\bar{a}_M \bar{a}_N \ldots} T^0_{j;a_M a_N \ldots}. \end{aligned} \tag{42}$$

and the indices i' denote other sites beside i and j.

In the last line we omit the minus sign because the two impurity tensors $\boldsymbol{T}'$ and $\boldsymbol{T}''$ contain *odd* number of Grassmann numbers hence they *anticommute* with each other.

In conclusion, the norm of a Grassmann tensor product state can be expressed as a tensor trace of uniform Grasmann tensor-net and other local physical quantities such as energy can be expressed as a tensor trace of Grassmann tensor-nets with impurity tensors. Here we have already seen that the fermion coherent state representation is extremely convenient for expressing the norm and physical quantities of Grassmann tensor product states as tensor traces of Grassmann tensor-nets. Essentially, such representations are the most natural representations for Grassmann tensor product state and provide us a deep insight into what is a fermion wavefunction. Actually, a fermion wavefunctions should be described Grassmann numbers rather than complex numbers, a detail discussions will be represented in our future publications.

VII. THE COARSE GRAINING TRANSFORMATION OF GRASSMANN TENSOR NETWORK

Expressing the norm and physical quantities of a strongly correlated wave function in terms of a tensor trace over a Grassmann tensor network is a very formal exercise. Calculating such a tensor trace directly is a exponential hard problem. So it seems that we gain nothing from writing the norm in the form of tensor trace.

However, if we only want to evaluate the norm and physical quantities approximately, then there is a polynomial way to do so in terms of the Grassmann tensor network. The basic idea is to perform a coarse graining transformation of the tensor network which can simplify the tensor network into one with only a few tensors.[22,24]

In this section, we will explain how to apply a coarse graining transformation to a Grassmann tensor network. We will discuss three basic moves on a honeycomb lattice, using rank-two, rank-three and rank-four tensors as examples.

In the first move, we combine two rank-three tensors $\boldsymbol{T}_i$, $\boldsymbol{T}_j$ and one rank-two tensor $\boldsymbol{G}_{ij}$ into one rank-four tensor $\boldsymbol{T}$ (see Fig. 5). We have

$$\boldsymbol{T}_{p_1p_2p_3p_4} = \sum_{p_5p_6} \int_{ij} \boldsymbol{T}_{i;p_5p_1p_2} \boldsymbol{T}_{j;p_6p_3p_4} \boldsymbol{G}_{ij;p_5p_6} \tag{43}$$

where $\int_{ij}$ only integrate over $\eta_i^{\alpha_5}$ and $\eta_j^{\alpha_6}$. If we expand $\boldsymbol{T}_i$, $\boldsymbol{T}_j$, $\boldsymbol{G}_{ij}$, and $\boldsymbol{T}_i$, we get

$$\begin{aligned}
\boldsymbol{T}_{i;p_5p_1p_2} &= \sum_{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}} \mathcal{T}_{i;p_5p_1p_2}^{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}} \widetilde{\prod_{\alpha_5}} (\mathrm{d}\eta_5^{\alpha_5})^{n_5^{\alpha_5}} \widetilde{\prod_{\alpha_1}} (\mathrm{d}\eta_1^{\alpha_1})^{n_1^{\alpha_1}} \widetilde{\prod_{\alpha_2}} (\mathrm{d}\eta_2^{\alpha_2})^{n_2^{\alpha_2}} \\
\boldsymbol{T}_{j;p_6p_3p_4} &= \sum_{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \mathcal{T}_{j;p_6p_3p_4}^{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \widetilde{\prod_{\alpha_6}} (\mathrm{d}\eta_6^{\alpha_6})^{n_6^{\alpha_6}} \widetilde{\prod_{\alpha_3}} (\mathrm{d}\eta_3^{\alpha_3})^{n_3^{\alpha_3}} \widetilde{\prod_{\alpha_4}} (\mathrm{d}\eta_4^{\alpha_4})^{n_4^{\alpha_4}} \\
\boldsymbol{T}_{p_1p_2p_3p_4} &= \sum_{\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \mathcal{T}_{p_1p_2p_3p_4}^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \widetilde{\prod_{\alpha_1}} (\mathrm{d}\eta_1^{\alpha_1})^{n_1^{\alpha_1}} \widetilde{\prod_{\alpha_2}} (\mathrm{d}\eta_2^{\alpha_2})^{n_2^{\alpha_2}} \widetilde{\prod_{\alpha_3}} (\mathrm{d}\eta_3^{\alpha_3})^{n_3^{\alpha_3}} \widetilde{\prod_{\alpha_4}} (\mathrm{d}\eta_4^{\alpha_4})^{n_4^{\alpha_4}} \\
\boldsymbol{G}_{ij;p_5p_6} &= \sum_{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}} \mathcal{G}_{ij;p_5p_6}^{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}} \prod_{\alpha_5} (\eta_5^{\alpha_5})^{n_5^{\alpha_5}} \prod_{\alpha_6} (\eta_6^{\alpha_6})^{n_6^{\alpha_6}}
\end{aligned} \tag{44}$$

We find that

$$\mathcal{T}_{p_1p_2p_3p_4}^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} = \sum_{p_5p_6} \sum_{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}} \mathcal{T}_{i;p_5p_1p_2}^{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}} \mathcal{G}_{ij;p_5p_6}^{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}} \mathcal{T}_{j;p_6p_3p_4}^{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \tag{45}$$

This allows us to calculate $\boldsymbol{T}$ from $\boldsymbol{T}_i$, $\boldsymbol{T}_j$, and $\boldsymbol{G}_{ij}$.

The second basic move splits the rank-four tensor $\boldsymbol{T}$ into two rank-three tensors $\boldsymbol{T}_i$, $\boldsymbol{T}_j$ and a rank-two tensor $\boldsymbol{G}_{ij}$ (see Fig. 6). We first rewrite $\mathcal{T}_{p_1p_2p_3p_4}^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}$ as (say, using the singular-value-decomposition(SVD) method discussed in Ref. 22)

$$\mathcal{T}_{p_1p_2p_3p_4}^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} = \sum_q \mathcal{S}_{i;qp_1p_2}^{\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}} \mathcal{S}_{j;qp_3p_4}^{\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}. \tag{46}$$

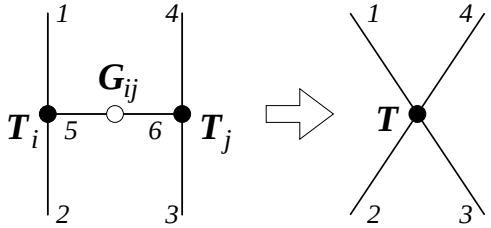

FIG. 5: We note that the Grassmann numbers $\theta_I^{\alpha_I}$ and $\mathrm{d}\theta_I^{\alpha_I}$ are associated with the links the connect the $\boldsymbol{G}$ tensors and $\boldsymbol{T}$ tensors. Here we label the links with $I = 1, 2, 3, 4, 5$, and 6. The tensor $\boldsymbol{T}_{i;p_5p_1p_2}$ contains Grassmann numbers $\{\mathrm{d}\eta_5^{\alpha_5}, \mathrm{d}\eta_1^{\alpha_1}, \mathrm{d}\eta_2^{\alpha_2}\}$. The tensor $\boldsymbol{T}_{j;p_6p_3p_4}$ contains Grassmann numbers $\{\mathrm{d}\eta_6^{\alpha_6}, \mathrm{d}\eta_3^{\alpha_3}, \mathrm{d}\eta_4^{\alpha_4}\}$. The tensor $\boldsymbol{T}$ contains Grassmann numbers $\{\mathrm{d}\eta_1^{\alpha_1}, \mathrm{d}\eta_2^{\alpha_2}, \mathrm{d}\eta_3^{\alpha_3}, \mathrm{d}\eta_4^{\alpha_4}\}$. The tensor $\boldsymbol{G}_{ij}$ contains Grassmann numbers $\{\eta_5^{\alpha_5}, \eta_6^{\alpha_6}\}$.

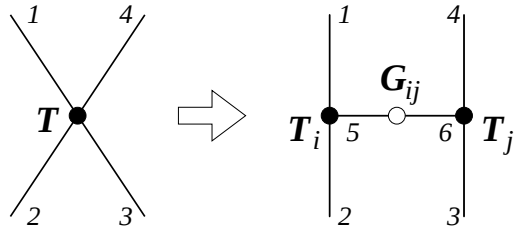

FIG. 6: Split one rank-four tensor into two rank-three tensors and a rank-two tensor.

The above decomposition can be rewritten as

$$\begin{aligned}&\mathcal{T}_{p_1p_2p_3p_4}^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}\\&=\sum_{p_5p_6}\mathcal{T}_{i;p_5p_1p_2}^{\{n_5\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}\mathcal{G}_{ij;p_5p_6}^{\{n_5\}\{n_6\}}\mathcal{T}_{j;p_6p_3p_4}^{\{n_6\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}\end{aligned}\tag{47}$$

where

$$\begin{aligned}\mathcal{G}_{ij;p_5p_6}^{\{n_5\}\{n_6\}}&=\delta_{p_5p_6}\delta_{n_5n_6},\\\mathcal{T}_{i;qp_1p_2}^{\{n_5\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}&=\mathcal{S}_{i;qp_1p_2}^{\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}},\\\mathcal{T}_{j;qp_3p_4}^{\{n_6\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}&=\mathcal{S}_{j;qp_3p_4}^{\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}},\end{aligned}\tag{48}$$

and

$$\begin{aligned}n_5&=\sum_{\alpha_1}n_1^{\alpha_1}+\sum_{\alpha_2}n_2^{\alpha_2}\ \mathrm{mod}\ 2\\n_6&=\sum_{\alpha_3}n_3^{\alpha_3}+\sum_{\alpha_4}n_4^{\alpha_4}\ \mathrm{mod}\ 2.\end{aligned}\tag{49}$$

We see that n_5 and n_6 are completely fixed by $n_I^{\alpha_I}$, $I = 1, 2, 3, 4$. Eqn. (48) define three Grassmann tensors (see (44))

$$\begin{aligned}\boldsymbol{T}_{i;p_5p_1p_2}&=\sum_{\{n_5\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}\mathcal{T}_{i;p_5p_1p_2}^{\{n_5\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}(\mathrm{d}\eta_5)^{n_5}\widetilde{\prod_{\alpha_1}}(\mathrm{d}\eta_1^{\alpha_1})^{n_1^{\alpha_1}}\widetilde{\prod_{\alpha_2}}(\mathrm{d}\eta_2^{\alpha_2})^{n_2^{\alpha_2}}\\\boldsymbol{T}_{j;p_6p_3p_4}&=\sum_{\{n_6\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}\mathcal{T}_{j;p_6p_3p_4}^{\{n_6\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}(\mathrm{d}\eta_6)^{n_6}\widetilde{\prod_{\alpha_3}}(\mathrm{d}\eta_3^{\alpha_3})^{n_3^{\alpha_3}}\widetilde{\prod_{\alpha_4}}(\mathrm{d}\eta_4^{\alpha_4})^{n_4^{\alpha_4}}\\\boldsymbol{G}_{ij;p_5p_6}&=(1+\eta_5\eta_6)\delta_{p_5p_6}\end{aligned}\tag{50}$$

This way, we split the tensor $\boldsymbol{T}$ into to the above three tensors, since $\boldsymbol{T}$ can be expressed in terms of the three tensors as in (43). It is interesting to note that $\boldsymbol{G}_{ij}$ reduces to a very simple form after one step of second move.

In the third move, we combine three rank-three tensors $\boldsymbol{T}_i$, $\boldsymbol{T}_j$, $\boldsymbol{T}_k$ and three rank-two tensors $\boldsymbol{G}_{ij}$, $\boldsymbol{G}_{jk}$, $\boldsymbol{G}_{ki}$ into one rank-three tensor $\boldsymbol{T}$ (see Fig. 7). If we expand the tensors $\boldsymbol{T}_{i;p_1p_4p_9}$, $\boldsymbol{T}_{j;p_2p_6p_5}$, $\boldsymbol{T}_{k;p_3p_8p_7}$ and $\boldsymbol{G}_{ij;p_4p_5}$, $\boldsymbol{G}_{jk;p_6p_7}$, $\boldsymbol{G}_{ki;p_8p_9}$, we obtain the following coefficients $\mathcal{T}_{i;p_1p_4p_9}^{\{n_1^{\alpha_1}\}\{n_4^{\alpha_4}\}\{n_9^{\alpha_9}\}}$, $\mathcal{T}_{j;p_2p_6p_5}^{\{n_2^{\alpha_2}\}\{n_6^{\alpha_6}\}\{n_5^{\alpha_5}\}}$, $\mathcal{T}_{k;p_3p_8p_7}^{\{n_3^{\alpha_3}\}\{n_8^{\alpha_8}\}\{n_7^{\alpha_7}\}}$, $\mathcal{G}_{ij;p_4p_5}^{\{n_4^{\alpha_4}\}\{n_5^{\alpha_5}\}}$, $\mathcal{G}_{jk;p_6p_7}^{\{n_6^{\alpha_6}\}\{n_7^{\alpha_7}\}}$, $\mathcal{G}_{ki;p_8p_9}^{\{n_8^{\alpha_8}\}\{n_9^{\alpha_9}\}}$. The coefficients of the resulting tensor $\boldsymbol{T}_{p_1p_2p_3}$ are given by

$$\begin{aligned}\mathcal{T}_{p_1p_2p_3}^{\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}}=&\sum_{p_4p_5p_6p_7p_8p_9}\sum_{\{n_4^{\alpha_4}\}}\sum_{\{n_5^{\alpha_5}\}}\sum_{\{n_6^{\alpha_6}\}}\sum_{\{n_7^{\alpha_7}\}}\sum_{\{n_8^{\alpha_8}\}}\sum_{\{n_9^{\alpha_9}\}}(-)^{(\sum_{\alpha_8}n_8^{\alpha_8})(\sum_{\alpha_9}n_9^{\alpha_9})}\times\\&\mathcal{T}_{i;p_1p_4p_9}^{\{n_1^{\alpha_1}\}\{n_4^{\alpha_4}\}\{n_9^{\alpha_9}\}}\mathcal{T}_{j;p_2p_6p_5}^{\{n_2^{\alpha_2}\}\{n_6^{\alpha_6}\}\{n_5^{\alpha_5}\}}\mathcal{T}_{k;p_3p_8p_7}^{\{n_3^{\alpha_3}\}\{n_8^{\alpha_8}\}\{n_7^{\alpha_7}\}}\mathcal{G}_{ij;p_4p_5}^{\{n_4^{\alpha_4}\}\{n_5^{\alpha_5}\}}\mathcal{G}_{jk;p_6p_7}^{\{n_6^{\alpha_6}\}\{n_7^{\alpha_7}\}}\mathcal{G}_{ki;p_8p_9}^{\{n_8^{\alpha_8}\}\{n_9^{\alpha_9}\}}\end{aligned}\tag{51}$$

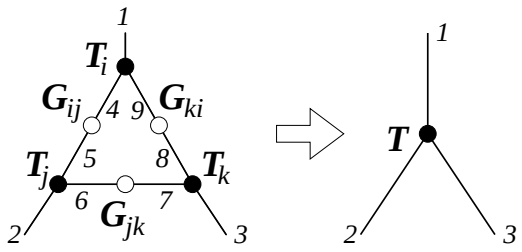

FIG. 7: Combine three rank-three tensors and three rank-two tensors into one rank-three tensor.

Just as in the bosonic TERG case[24], the above coarse graining transformation can also be generalized onto other plainer graphs, such as square, kagome, triangular lattice, etc. However, on generic graphs, especially those graphs with long range connections, the tensor contraction still can be exponentially hard in most cases.

VIII. EXAMPLE: FREE FERMION STATE ON A HONEYCOMB LATTICE

To test the coarse graining procedure for the Grassmann tensor network discussed above, let us study a simple example of a free fermion state and its tensor network representations. We assume fermions live on the vertices of a honeycomb lattice. Let us consider the pairing state

$$|\Psi_f\rangle = \mathrm{e}^P|0\rangle \tag{52}$$

on a honeycomb lattice. Here the pairing operator is given by

$$P = \sum_{\langle \boldsymbol{ij}\rangle} u c^\dagger_{\boldsymbol{i}} c^\dagger_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in A, a=1,2,3} u c^\dagger_{\boldsymbol{i}} c^\dagger_{\boldsymbol{i}+\boldsymbol{\delta}_a}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ label sites and $\langle \boldsymbol{ij}\rangle$ labels the nearest-neighbor links of the honeycomb lattice. We know that the sites of the honeycomb lattice can be divided into two sub-lattice: A and B (see Fig. 8). The pairing of the fermions is only between the two different sublattices. Here we have used the convention that in the link label $\langle \boldsymbol{ij}\rangle$, $\boldsymbol{i}$ be long to the A-sublattice and $\boldsymbol{j}$ belong to the B-sublattice. The three vectors $\boldsymbol{\delta}_a$, $a = 1, 2, 3$, are the three vectors that connect a A-site to to its three nearest neighboring B-sites.

Introduce

$$c_A(\boldsymbol{k}) = \sqrt{\frac{2}{N}} \sum_{\boldsymbol{i}\in A} \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{i}} c_{\boldsymbol{i}}$$

$$c_B(\boldsymbol{k}) = \sqrt{\frac{2}{N}} \sum_{\boldsymbol{i}\in B} \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{i}} c_{\boldsymbol{i}} = \sqrt{\frac{2}{N}} \sum_{\boldsymbol{i}\in A} \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{i}+\boldsymbol{\delta}_a)} c_{\boldsymbol{i}+\boldsymbol{\delta}_a}$$

where N is the total number of lattice sites, $\sum_{\boldsymbol{i}\in A}$ sums over the sites in the A-sublattice and $\sum_{\boldsymbol{i}\in B}$ over the B-sublattice. We can rewrite P as

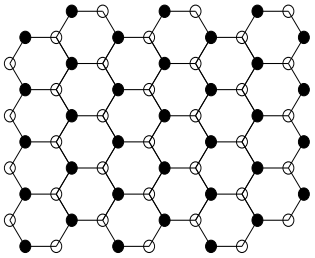

FIG. 8: A honeycomb lattice and its two sublattices. The distance between two nearest neighboring sites is chosen to be 1.

$$P = \sum_{\boldsymbol{k}} c^\dagger_A(-\boldsymbol{k}) \alpha_{\boldsymbol{k}} c^\dagger_B(\boldsymbol{k})$$

with

$$\alpha_{\boldsymbol{k}} = u[\mathrm{e}^{-\mathrm{i}\,k_x} + \mathrm{e}^{-\mathrm{i}\,(-\frac{\sqrt{3}k_y}{2} - \frac{k_x}{2})} + \mathrm{e}^{-\mathrm{i}\,(\frac{\sqrt{3}k_y}{2} - \frac{k_x}{2})}].$$

Therefore

$$|\Psi_f\rangle = \prod_{\boldsymbol{k}} [1 + \alpha_{\boldsymbol{k}} c^\dagger_A(-\boldsymbol{k}) c^\dagger_B(\boldsymbol{k})]|0\rangle. \tag{53}$$

Let us rewrite

$$\begin{aligned} &[1 + \alpha_{\boldsymbol{k}} c^\dagger_A(-\boldsymbol{k}) c^\dagger_B(\boldsymbol{k})]|0\rangle \\ &\propto [v_{\boldsymbol{k}} c^\dagger_A(-\boldsymbol{k}) + u_{\boldsymbol{k}} c_B(\boldsymbol{k})][u_{\boldsymbol{k}} c_A(-\boldsymbol{k}) - v_{\boldsymbol{k}} c^\dagger_B(\boldsymbol{k})]|0\rangle \end{aligned} \tag{54}$$

where

$$|u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1, \quad \alpha_{\boldsymbol{k}} = v_{\boldsymbol{k}}/u_{\boldsymbol{k}}. \tag{55}$$

We find

$$v_{\boldsymbol{k}} = \frac{\alpha_{\boldsymbol{k}}}{\sqrt{1 + |\alpha_{\boldsymbol{k}}|^2}}, \quad u_{\boldsymbol{k}} = \frac{1}{\sqrt{1 + |\alpha_{\boldsymbol{k}}|^2}}. \tag{56}$$

Let

$$\begin{aligned} \psi_{1,-\boldsymbol{k}} &= [u_{\boldsymbol{k}} c_A(-\boldsymbol{k}) - v_{\boldsymbol{k}} c^\dagger_B(\boldsymbol{k})], \\ \psi_{2,\boldsymbol{k}} &= [v_{\boldsymbol{k}} c^\dagger_A(-\boldsymbol{k}) + u_{\boldsymbol{k}} c_B(\boldsymbol{k})], \end{aligned} \tag{57}$$

which satisfy the standard commutation relation for fermion operators. By construction, we see that

$$\psi_{1,\boldsymbol{k}}|\Phi_f\rangle = \psi_{2,\boldsymbol{k}}|\Phi_f\rangle = 0. \tag{58}$$

Thus $|\Psi_f\rangle$ is the ground state of the following quadratic

Hamiltonian

$$
\begin{aligned}
H &= \sum_{\boldsymbol{k}} (1+|\alpha_{\boldsymbol{k}}|^2)(\psi^\dagger_{1,\boldsymbol{k}}\psi_{1,\boldsymbol{k}} + \psi^\dagger_{2,\boldsymbol{k}}\psi_{2,\boldsymbol{k}}) \\
&= \sum_{\boldsymbol{k}} [c_A^\dagger(-\boldsymbol{k}) - \alpha^*_{\boldsymbol{k}} c_B(\boldsymbol{k})][c_A(-\boldsymbol{k}) - \alpha_{\boldsymbol{k}} c_B^\dagger(\boldsymbol{k})] \\
&\quad + \sum_{\boldsymbol{k}} [\alpha^*_{\boldsymbol{k}} c_A(-\boldsymbol{k}) + c_B^\dagger(\boldsymbol{k})][\alpha_{\boldsymbol{k}} c_A^\dagger(-\boldsymbol{k}) + c_B(\boldsymbol{k})] \\
&= \sum_{\boldsymbol{k}} \Big[-2c_A^\dagger(-\boldsymbol{k})\alpha_{\boldsymbol{k}} c_B^\dagger(\boldsymbol{k}) + h.c.\Big] \\
&\quad + \sum_{\boldsymbol{k}} (1-|\alpha_{\boldsymbol{k}}|^2)[c_A^\dagger(\boldsymbol{k})c_A(\boldsymbol{k}) + c_B^\dagger(\boldsymbol{k})c_B(\boldsymbol{k})] \\
&\quad + \sum_{\boldsymbol{k}} 2|\alpha_{\boldsymbol{k}}|^2. \qquad (59)
\end{aligned}
$$

In real space, the above Hamiltonian can be rewritten as

$$
\begin{aligned}
H &= \sum_{\boldsymbol{k}} 2|\alpha_{\boldsymbol{k}}|^2 - \sum_{\langle \boldsymbol{ij}\rangle} (2uc_{\boldsymbol{i}}^\dagger c_{\boldsymbol{j}}^\dagger + h.c.) \\
&\quad + \sum_{\boldsymbol{i}} (1-3|u|^2) c_{\boldsymbol{i}}^\dagger c_{\boldsymbol{i}} - \sum_{\boldsymbol{i},I=1,\ldots,6} |u|^2 c^\dagger_{\boldsymbol{i}+\boldsymbol{\Delta}_I} c_{\boldsymbol{i}} \qquad (60)
\end{aligned}
$$

where $\{\boldsymbol{\Delta}_I\}$ are six vectors $\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2$, $-\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2$, $\boldsymbol{\delta}_2 - \boldsymbol{\delta}_3$, $-\boldsymbol{\delta}_2 + \boldsymbol{\delta}_3$, $\boldsymbol{\delta}_3 - \boldsymbol{\delta}_1$, and $-\boldsymbol{\delta}_3 + \boldsymbol{\delta}_1$. Note that, if we do a particle-hole conjugation on the B-sublattice, the above pairing Hamiltonian becomes the following hopping Hamiltonian

$$
\begin{aligned}
H &= -\sum_{\langle \boldsymbol{ij}\rangle} (2uc_{\boldsymbol{i}}^\dagger c_{\boldsymbol{j}} + h.c.) - \sum_{\boldsymbol{i},I=1,\ldots,6} (-)^{\boldsymbol{i}} |u|^2 c^\dagger_{\boldsymbol{i}+\boldsymbol{\Delta}_I} c_{\boldsymbol{i}} \\
&\quad + \sum_{\boldsymbol{i}} (1-3|u|^2)(-)^{\boldsymbol{i}} c_{\boldsymbol{i}}^\dagger c_{\boldsymbol{i}} + \text{Const.} \qquad (61)
\end{aligned}
$$

where $(-)^{\boldsymbol{i}} = 1$ if $\boldsymbol{i}$ is in A-sublattice and $(-)^{\boldsymbol{i}} = -1$ if $\boldsymbol{i}$ is in B-sublattice.

Fig. 9 shows the expectation values of the nearest paring term $c_{\boldsymbol{i}}^\dagger c_{\boldsymbol{j}}^\dagger$, next nearest neighbor hopping term $c^\dagger_{\boldsymbol{i}+\boldsymbol{\Delta}_I} c_{\boldsymbol{i}}$ as well as onsite fermion number term $c_{\boldsymbol{i}}^\dagger c_{\boldsymbol{i}}$ calculated from the Grassmann-number tensor renormalization algorithm. We found very good agreement with the exact results are found for a large range of parameter u. The SVD truncation dimension is set to be $D_{cut} = 32$ throughout the whole calculation. Fig. 10 shows the D_{cut} dependence for the physical quantities we calculated. We find excellent agreement with exact results for large enough D_{cut}.

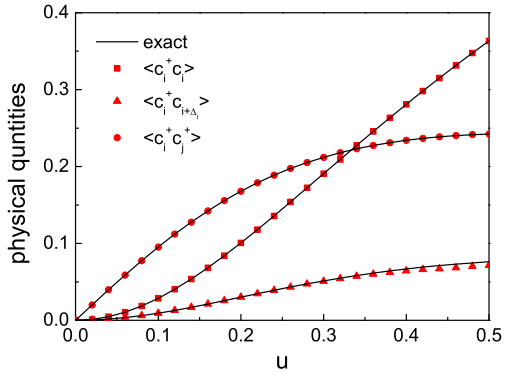

FIG. 9: The expectation value of the nearest neighbor paring, next nearest neighbor hopping and onsite fermion number terms from the Grassmann-number tensor-entanglement renormalization algorithm. We compare our calculation with the exact results and find a good agreement for a large range of u.

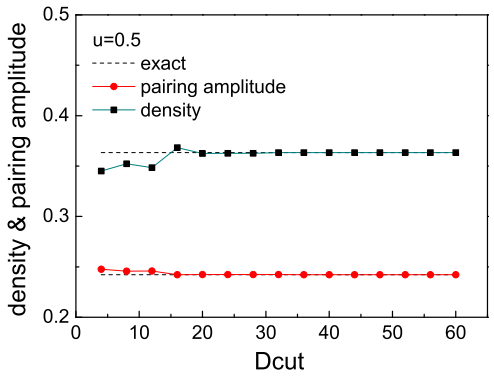

FIG. 10: The expectation value of the nearest neighbor paring and onsite fermion number at different D_{cut}(the truncation dimension in the SVD decomposition) in the Grassmann-number tensor-entanglement renormalization algorithm. It shows all those values converge to the exact one at sufficient large D_{cut}. In this calculation we choose $u = 0.5$.

IX. SUMMARY

The projective construction has played an very important role in understanding strongly correlated systems. It can explain the emergence of fermions, anyons (Abelian and non-Abelian), and gauge theory in quantum spin liquids and quantum Hall states. Mathematically, the projective construction also provides an efficient encoding for many-body bosonic/fermionic states.

Recently, Fermionic projected entangled pair states have also been introduced as an alternative method for efficiently encoding many-body fermionic states. In this paper, we show that the strongly correlated bosonic/fermionic states obtained from the projective construction and fPEPS approach can be represented systematically as Grassmann tensor product states. These Grassmann tensor product states allow us to encode many-body bosonic/fermionic states with a number of parameters that only scales polynomially in the number of fermions. We have also shown that it is possible to generalize the tensor-entanglement renormaliza-

tion group (TERG) method for complex tensor networks to Grassmann tensor networks. This allows us to perform an approximate calculation of the norm and average local operators of Grassmann tensor product states in polynomial time.

In conclusion, Grassmann tensor product states can be a starting point for a new variational approach to strongly correlated bosonic/fermionic systems, as they not only include tensor product state and fermion ELF states, but also systematically generalize the slave-particle projective construction. All the physical properties of these states can be efficiently calculated based on the Grassmann tensor-entanglement renormalization group (GTERG) algorithm. Many non-trivial fermionic models and frustrated spin models, such as t-J model, Hubbard model, as well as Kagome Heisenberg model will be studied in our future work.

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Appendix A: Calculate the physical measurements

In this section, we explain how to use coarse graining transformation to calculate the physical quantities of a Grassmann tensor product state. As a simple example, we first explain how to calculate the expectation value for nearest neighbor electron pairing term $\langle\Psi|c_i^\dagger c_j^\dagger|\Psi\rangle$. Such a term can be represented as a Grassmann tensor network with two impurity tensors, see Eq. (41).

We can apply the coarse graining transformation for the uniform part in the same way as we calculate the norm. The extra thing we need to know is how to apply the coarse graining transformation for the two impurity tensors. It turns out we only need to modify the first step, where we combine the two rank three impurity tensors into a rank four impurity tensor

$$\boldsymbol{T}'''_{p_1p_2p_3p_4} = \sum_{p_5p_6}\int_{ij} \boldsymbol{T}'_{i;p_5p_1p_2}\boldsymbol{T}''_{j;p_6p_3p_4}\boldsymbol{G}_{ij;p_5p_6}, \quad \text{(A1)}$$

with

$$\begin{aligned}
\boldsymbol{T}'_{i;p_5p_1p_2} &= \sum_{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}} \mathcal{T}'^{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}_{i;p_5p_1p_2}\widetilde{\prod_{\alpha_5}}(\mathrm{d}\eta_5^{\alpha_5})^{n_5^{\alpha_5}}\widetilde{\prod_{\alpha_1}}(\mathrm{d}\eta_1^{\alpha_1})^{n_1^{\alpha_1}}\widetilde{\prod_{\alpha_2}}(\mathrm{d}\eta_2^{\alpha_2})^{n_2^{\alpha_2}}\\
\boldsymbol{T}''_{j;p_6p_3p_4} &= \sum_{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \mathcal{T}''^{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}_{j;p_6p_3p_4}\widetilde{\prod_{\alpha_6}}(\mathrm{d}\eta_6^{\alpha_6})^{n_6^{\alpha_6}}\widetilde{\prod_{\alpha_3}}(\mathrm{d}\eta_3^{\alpha_3})^{n_3^{\alpha_3}}\widetilde{\prod_{\alpha_4}}(\mathrm{d}\eta_4^{\alpha_4})^{n_4^{\alpha_4}}\\
\boldsymbol{T}'''_{p_1p_2p_3p_4} &= \sum_{\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}} \mathcal{T}'''^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}_{p_1p_2p_3p_4}\widetilde{\prod_{\alpha_1}}(\mathrm{d}\eta_1^{\alpha_1})^{n_1^{\alpha_1}}\widetilde{\prod_{\alpha_2}}(\mathrm{d}\eta_2^{\alpha_2})^{n_2^{\alpha_2}}\widetilde{\prod_{\alpha_3}}(\mathrm{d}\eta_3^{\alpha_3})^{n_3^{\alpha_3}}\widetilde{\prod_{\alpha_4}}(\mathrm{d}\eta_4^{\alpha_4})^{n_4^{\alpha_4}}\\
\boldsymbol{G}_{ij;p_5p_6} &= \sum_{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}} \mathcal{G}^{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}}_{ij;p_5p_6}\prod_{\alpha_5}(\eta_5^{\alpha_5})^{n_5^{\alpha_5}}\prod_{\alpha_6}(\eta_6^{\alpha_6})^{n_6^{\alpha_6}}.
\end{aligned} \quad \text{(A2)}$$

Because $\boldsymbol{T}'$ and $\boldsymbol{T}''$ contain an odd number of Grassmann numbers in this case, the relationship of the coefficients should be modified as:

$$\mathcal{T}'''^{\{n_2^{\alpha_2}\}\{n_2^{\alpha_2}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}_{p_1p_2p_3p_4} = \sum_{p_5p_6}\sum_{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}}(-)^{(\sum_{\alpha_5} n_5^{\alpha_5})}\mathcal{T}'^{\{n_5^{\alpha_5}\}\{n_1^{\alpha_1}\}\{n_2^{\alpha_2}\}}_{i;p_5p_1p_2}\mathcal{G}^{\{n_5^{\alpha_5}\}\{n_6^{\alpha_6}\}}_{ij;p_5p_6}\mathcal{T}''^{\{n_6^{\alpha_6}\}\{n_3^{\alpha_3}\}\{n_4^{\alpha_4}\}}_{j;p_6p_3p_4}. \quad \text{(A3)}$$

The second step is the same as we calculate the norm because the rank four impurity tensor contain an *even* number of Grassmann numbers. Of course the last step also remains the same except we need to combine one new impurity tensor with two new uniform tensors to produce a coarse grained impurity tensor, as we do in the standard TERG algorithm, see in Fig. 11. If we calculate the expectation value for two nearest neighbor bosonic operators, such as $\langle n_i n_j\rangle$, even the first step dose not need to be modified because the impurity tensors contain even number of Grassmann numbers in this case.

For generic interactions as well as correlation functions, the coarse graining procedure can be designed in the same way as in the TERG algorithm. Fig. 12 shows how to do the coarse graining transformation for generic six body interactions on the hexagon of a honeycomb lattice. However, we need to take care of the sign factor when we apply the coarse graining transformation for

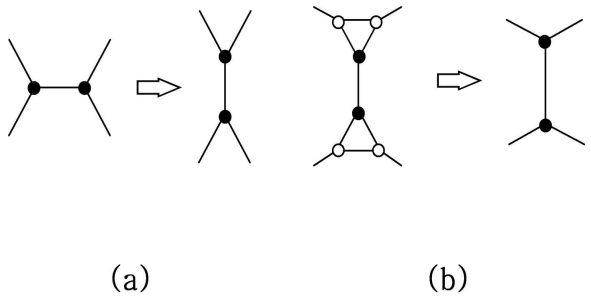

FIG. 11: A schematic plot of how to apply coarse graining transformations for two nearest neighbor impurity tensors on a honeycomb lattice. The filled dots represent impurity tensors and open dots represent uniform tensors. In (a), we first combine two rank three impurity tensors into a rank four impurity tensor and then split it into two new rank three impurity tensors. In (b), we apply the last step to combine one impurity tensor with two uniform tensors to produce a new impurity tensor.

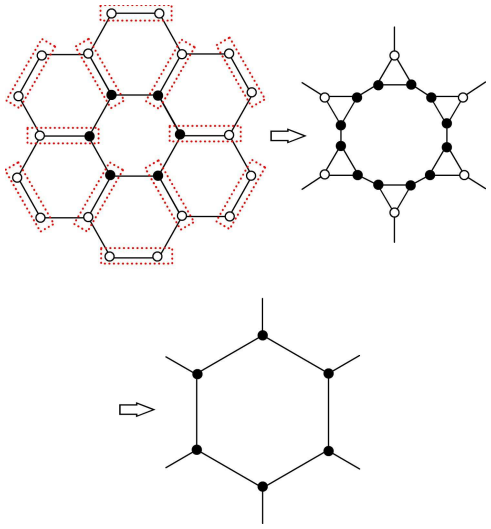

FIG. 12: A schematic plot of the coarse graining transformations for generic six body interactions on the hexagon of a honeycomb lattice. The filled dots represent impurity tensors and open dots represent uniform tensors.

impurity tensors which contain an odd number of Grassmann number and in the most generic case, all the three steps need to be modified. For example, in the second step, if the rank four impurity tensor $\boldsymbol{T}'''$ contain an odd number of Grassmann numbers, then we need to decompose it into two new rank three impurity tensors where one has an odd number of Grassmann numbers and the other has an even number of Grassmann numbers. We also need to take care of the sign factor in the last step if we combine impurity tensors with odd number of Grassmann numbers.

Appendix B: A review of projective construction for fractional quantum Hall state

In this section, we are going to review two examples of fractional quantum Hall state to demonstrate how to obtain low energy effective theory from a projective approach.[8,10]

To use the projective construction to study the $\nu = 1/3$ Laughlin state of N electrons, we consider a system of three kinds of partons described by the fermion operators $\psi_a(z)$, $a = 1, 2, 3$. The parton system contain N particles for each kind of partons, and each kind of parton form $\nu = 1$ state (which is a ELF state). Let us denote the $\nu = 1$ state for the a^{th} partons as $|\Psi_1\rangle_a$. Then the total ground state of the system of three kinds of partons is given by $|\Psi_1\rangle_1 \otimes |\Psi_1\rangle_2 \otimes |\Psi_1\rangle_3 \equiv |\Psi_1\Psi_1\Psi_1\rangle$. Note that the state $|\Psi_1\Psi_1\Psi_1\rangle$ is a ELF state. Then the $\nu = 1/3$ Laughlin wave function can be expressed as a projection of the ELF state $|\Psi_1\Psi_1\Psi_1\rangle$:

$$\Psi_3(\{z_i\}) = \langle 0| \prod_i [\psi_1(z_i)\psi_2(z_i)\psi_3(z_i)] |\Psi_1\Psi_1\Psi_1\rangle. \quad \text{(B1)}$$

Note that the projection simply combine the three partons into a single electron:

$$c(z) = \psi_1(z)\psi_2(z)\psi_3(z). \quad \text{(B2)}$$

Although the expression (B1) is very formal, the projection that it describes can be done at the field theory level, which allows us to calculate the low energy effective theory of the Laughlin state.

To do the projection at the field theory level, we start with the Lagrangian that describes three independent partons (that is before the projection)

$$\mathcal{L}_0 = \sum_{a=1}^{3} \Big[\psi_a^\dagger \mathrm{i}\partial_t \psi_a - \frac{1}{2m} |(\boldsymbol{\partial} - \mathrm{i}\frac{e}{3}\boldsymbol{A})\psi_a|^2 - \mu \psi_a^\dagger \psi_a \Big], \quad \text{(B3)}$$

where we have assumed that each kind of parton carries $e/3$ electric charge and $\boldsymbol{A}$ is the vector potential that describes the uniform magnetic field. The chemical potential is chosen such that each kind of parton fills its first Landau level and forms the $\nu = 1$ quantum Hall state.

We note that the above Lagrangian that describes the independent partons before the projection has an $SU(3)$ symmetry:

$$\psi_a \to U_{ab}\psi_b, \quad U \in SU(3). \quad \text{(B4)}$$

The theory of independent partons contains fluctuations of the density and the current of the $SU(3)$ charge. On the other hand, we see that the electron operator c transforms as

$$c = \psi_1\psi_2\psi_3 \to \det(U)\psi_1\psi_2\psi_3 = c \quad \text{(B5)}$$

since the ψ_a anticommute with each other. Thus the electron operator is invariant under the $SU(3)$ transformation. As a result, the electronic state obtained after

the projection in (B1) is also invariant under the $SU(3)$ transformation since both $|0\rangle$ and c are $SU(3)$ invariant. This means that, after the projection, the electronic state contains no fluctuations of the density and the current of the $SU(3)$ charges.

This motivates us to perform the projection at the field theory level by including a $SU(3)$ gauge theory in the above independent-parton model:

$$\begin{aligned}\mathcal{L}_p = &\sum_{a,b} \psi_a^\dagger \mathrm{i}[\delta_{ab}\partial_t - \mathrm{i}(a_0)_{ab}]\psi_b \\ &+ \frac{1}{2m}\sum_{a,b}\psi_a^\dagger(\boldsymbol{\partial} - \mathrm{i}\frac{e}{3}\boldsymbol{A} - \mathrm{i}\boldsymbol{a})_{ab}^2\psi_b - \mu\sum_a \psi_a^\dagger\psi_a,\end{aligned} \tag{B6}$$

where (a_0, a_x, a_y) are the $SU(3)$ gauge fields which are 3 by 3 hermitian matrix valued fields. The $SU(3)$ gauge fields remove all the $SU(3)$ density and current fluctuations. In other word, if we perform the path integral of the $SU(3)$ gauge fields first

$$\mathrm{e}^{-\mathrm{i}\int dtd^2\boldsymbol{x}\,\mathcal{L}_e} = \int D[\boldsymbol{a}]D[a_0]\,\mathrm{e}^{-\mathrm{i}\int dtd^2\boldsymbol{x}\,\mathcal{L}_p}, \tag{B7}$$

the resulting effective theory $\mathcal{L}_e$ will contain no $SU(3)$ fluctuations. Thus we say that the path integral of the $SU(3)$ gauge fields, $\int D[\boldsymbol{a}]D[a_0]$, performs the projection at the field theory level.

The full theory is described by the path integral over both parton fields ψ_a and $SU(3)$ gauge fields (a_0, a_x, a_y):

$$Z = \int D[\psi_a]\int D[\boldsymbol{a}]D[a_0]\,\mathrm{e}^{-\mathrm{i}\int dtd^2\boldsymbol{x}\,\mathcal{L}_p} \tag{B8}$$

If we exchange the integration order:

$$\begin{aligned}Z &= \int D[\boldsymbol{a}]D[a_0]\int D[\psi_a]\,\mathrm{e}^{-\mathrm{i}\int dtd^2\boldsymbol{x}\,\mathcal{L}_p} \\ &= \int D[\boldsymbol{a}]D[a_0]\,\mathrm{e}^{-\mathrm{i}\int dtd^2\boldsymbol{x}\,\mathcal{L}_a},\end{aligned} \tag{B9}$$

we obtain an effective theory that contain only the $SU(3)$ gauge field $\mathcal{L}_a(a_0, a_x, a_y)$. Since each kind of parton forms the $\nu = 1$ quantum Hall state, the resulting $SU(3)$ effective theory turns out to be the level-1 $SU(3)$ Chern-Simons theory. We conclude that the $\nu = 1/3$ Laughlin state is described by the level-1 $SU(3)$ Chern-Simons topological field theory. All the topological properties of the $\nu = 1/3$ Laughlin state, such as fractional charges and fractional statistics can be obtained from such a $SU(3)_1$ Chern-Simons theory. Note that the level-1 $SU(3)$ Chern-Simons theory is equivalent to a $U(1)$ Chern-Simons theory. Thus $SU(3)_1$ Chern-Simons theory actually describes an Abelian state.

We have seen that if we let partons to form the $\nu = 1$ quantum Hall state, we will obtain the $\nu = 1/3$ Laughlin state and the $SU(3)_1$ Chern-Simons theory as its low energy effective theory. If we let each of the three kinds of partons forms the $\nu = m$ quantum Hall state, we will obtain a $\nu = m/3$ quantum Hall state and the level-m $SU(3)$ Chern-Simons theory as its low energy effective theory.[10] Such a $\nu = m/3$ quantum Hall state is a non-Abelian quantum Hall state.

Similarly, starting from the ELF state for four kinds of partons $|\Psi_1\Psi_1\Psi_1\Psi_1\rangle$, we can construct the non-Abelian $\nu = 1$ Pfaffian state (for bosonic electrons)

$$\begin{aligned}&\Psi_{\mathrm{Pfa}}(\{z_i\}) \\ &= \langle 0|\prod_i[\psi_1(z_i)\psi_4(z_i) - \psi_3(z_i)\psi_2(z_i)]|\Psi_1\Psi_1\Psi_1\Psi_1\rangle.\end{aligned} \tag{B10}$$

where the electron operator is related to the parton operators as

$$c(z) = \psi_1(z)\psi_4(z) - \psi_3(z)\psi_2(z). \tag{B11}$$

To do the projection at the field theory level, we start with the independent parton model

$$\mathcal{L}_0 = \sum_{a=1}^4\Big[\psi_a^\dagger \mathrm{i}\partial_t\psi_a - \frac{1}{2m}|(\boldsymbol{\partial} - \mathrm{i}\frac{e}{2}\boldsymbol{A})\psi_a|^2 - \mu\psi_a^\dagger\psi_a\Big], \tag{B12}$$

We note that the above Lagrangian has a $SU(4)$ symmetry:

$$\psi_a \to U_{ab}\psi_b, \quad U \in SU(4). \tag{B13}$$

Thus the theory of independent partons contains $SU(4)$ charge and current fluctuations.

However, the electron operator $c = \psi_1\psi_4 - \psi_3\psi_2$ is not invariant under the full $SU(4)$ transformations. If we identify $(\psi_1, ..., \psi_4) = (\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$, we find

$$\begin{aligned}c &= \psi_{11}\psi_{22} - \psi_{21}\psi_{12} \\ &\propto \psi_{11}\psi_{22} - \psi_{21}\psi_{12} + \psi_{12}\psi_{21} - \psi_{22}\psi_{11} \\ &\propto \psi_{a\alpha}(\tau_2)_{ab}(\sigma_1)_{\alpha\beta}\psi_{b\beta} \propto \psi^T(\tau_2\otimes\sigma_2)(\tau_0\otimes\sigma_3)\psi,\end{aligned} \tag{B14}$$

where $\tau_0 = \sigma_0$ are the 2 by 2 identity matrix and τ_l, σ_l are the Pauli matrices. Here τ_l acts on the first subscript a of $\psi_{a\alpha}$ while σ_l acts on the second subscript α of $\psi_{a\alpha}$. We see that the electron operator is invariant under a subgroup of $SU(4)$ generated by 10 generators: $\tau_i\otimes\sigma_0$, $\tau_i\otimes\sigma_1$, $\tau_i\otimes\sigma_2$, and $\tau_0\otimes\sigma_3$, It turns out that the above 10 generators generate the $SO(5)$ group in its 4 dimensional spinor representation. Therefore, the electronic states do not contain any $SO(5)$ fluctuations.

To remove the $SO(5)$ fluctuations at the field theory level, we can include a $SO(5)$ gauge field (the 4 dimensional representation spanned by $\tau_i\otimes\sigma_0$, $\tau_i\otimes\sigma_1$, $\tau_i\otimes\sigma_2$, and $\tau_0\otimes\sigma_3$) in the parton Lagrangian. After integrating out the partons we obtain the low energy effective Chern-Simons theory for the Pfaffian state, which is a $SO(5)$ Chern-Simons theory. A different effective Chern-Simons theory for the Pfaffian state was obtained in Ref. 28.

We would like to point out that in the first form of the projective construction (B1), the electron operator $c(z)$ is expressed as a product of parton operators. In this

case, we can use variational Monte Carlo calculation to numerically study many properties (such as ground state energy) of such projective states. On the other hand, in the second form of the projective construction (B10), the electron operator $c(z)$ is expressed as a sum of several products of parton operators. In this case, in general, the variational Monte Carlo method is ineffective due to the sign problem. So far, we still do not have an effective numerical method for the second form of the projective construction.

The Grassmann tensor network can represent projective states obtained from the both forms of the projective construction. So the renormalization of the Grassmann tensor network might allow us to approximately calculate the norms and average local operators for both forms of projective states. A detailed study will be presented in our future work.

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